Decrease of entanglement by local operations in the Dür–Cirac method

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One cannot always obtain information about entanglement by the Dür–Cirac (DC) method. The impracticality is attributed to the decrease of entanglement by local operations in the DC method. We show that, even in 2–qubit systems, there exist states whose entangled property the DC method never evaluates. The class of such states in 2–qubit systems is completely characterized by the value of the fully entangled fraction. Actually, a state whose fully entangled fraction is less than or equal to $\frac{1}{2}$ is always transformed into a separable state by local operations in the DC method, even if it has negative partial transposition.

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I. INTRODUCTION

Quantum mechanics has a quite different mathematical and conceptual structure from that of classical mechanics. Quantum entanglement vividly illustrates this point [1]. Investigation into the character of entanglement is necessary for not only the deep understanding of quantum theory but also its application. Indeed, entanglement is regarded as a key concept of quantum information processing [2].

The classification and quantification of bipartite entanglement (i.e., entanglement between two subsystems in a total quantum system) are well established [3, 4, 5, 6, 7, 8, 9, 10]. In particular, the positive partial transposition criterion (PPT) [3, 4, 5, 6] is very useful, because one can readily obtain a sufficient condition for an entangled state, a necessary condition for a separable state (i.e., a state with no quantum correlation) [11], or a necessary condition for a distillable state [10], by linear algebra.

The situation becomes more complicated as the number of subsystems in a total system increases. In 3–qubit systems, for example, there are two inequivalent classes of entanglement. By stochastic local operations and classical communication [12], a Greenberger–Horn–Zeilinger (GHZ) state cannot be transformed into a W state, and vice versa [13]. However, multiparticle entanglement can play an important role in quantum protocol (e.g., quantum telecloning [14]) and quantum computing. Moreover, its classification will be useful for deeply understanding quantum phase transitions in condensed matter physics [15]. Thus, research into multiparticle entanglement is a crucial and popular issue in both quantum physics and quantum information theory.

Various attempts to classify and quantify multiparticle entanglement have been made [16, 17, 18, 19, 20, 21, 22, 23]. Among them, Dür and Cirac [17] proposed a systematic way of classifying multiparticle entanglement in N–qubit systems. Hereafter, we call it the Dür–Cirac (DC) method. The main idea is that, using a sequence of local operations, one can transform an arbitrary density matrix of an N–qubit system into a state whose entangled property is easily examined. It should be noted that entanglement cannot increase through local operations. Accordingly, if the density matrix transformed by local operations is entangled, then the original density matrix represents an entangled state.

However, one cannot always obtain an entangled property by the DC method. In our previous paper [24], we suggested that there exists an impracticality in the DC method through an example.

In this letter, we reveal the possibility that one cannot obtain the desired information on entanglement by the DC method, though it is a very simple and effective method for examining multiqubit entanglement. We show that there is such a possibility even in 2–qubit systems. The most important quantity in our discussion is a fully entangled fraction [7, 10]. Our main result is that, in 2–qubit systems, one can never determine whether a quantum state is entangled or not through the DC method if the fully entangled fraction is less than or equal to $\frac{1}{2}$. Then, we completely characterize the class of the state in 2–qubit systems whose entangled property is never obtained by the DC method. The impracticality of the DC method is due to the decrease of entanglement by local operations in the DC method. Additionally, we investigate what parts of the local operations reduce the entanglement in 2–qubit system.

The letter is organized as follows. We briefly review the DC method in section II. Then, we illustrate the impracticality of the DC method through an example in a 2–qubit system, and show the relation to the local operations in section III. After that, we show our main results in section IV. Our results are shown only in 2–qubit systems, but they clearly reveal the limitation of the DC method. Section V is devoted to a summary.

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II. REVIEW OF THE DC METHOD

We briefly review the DC method [16, 17, 18, 19]. Its main idea is that, using a sequence of local operations, one can transform an arbitrary density matrix of an N–qubit system into a state whose property of entanglement is easily examined.

First, we explain how to specify a bipartition of the system concerned. We divide an N–qubit system into two subsystems, system A and system B, as follows. Let us consider a set of binary numbers, $\{k_i\}_{i=1}^{N}$ ($k_i = 0, 1$). When k_i is equal to 0 (1), the ith qubit is in system A (B). We always set $k_1 = 0$; the first qubit is always in system A. Representing the number by a binary, $k \equiv \sum_{i=2}^{N} k_i 2^{i-2}$, a partition is specified in the N–qubit system if an integer $k(\in [1,\, 2^{N-1}-1])$ is chosen; we call such a partition the bipartition k.

The authors in Ref. [16, 17, 18, 19] introduced a special family of density matrices as follows:

$$\rho_N = \lambda_0^+ |\Psi_0^+\rangle\langle\Psi_0^+| + \lambda_0^- |\Psi_0^-\rangle\langle\Psi_0^-| + \sum_{j=1}^{2^{N-1}-1} \lambda_j \big(|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|\big), \quad (1)$$

where the coefficients $\lambda_0^{\pm}$ and λ_j are real and positive, and $\lambda_0^+ + \lambda_0^- + \sum_{j=1}^{2^{N-1}-1} \lambda_j = 1$ because $\mathrm{tr}\rho_N = 1$. These coefficients are related to the information on an arbitrary density matrix of an N–qubit system, as shown below. The generalized GHZ state [16, 17, 18, 19] in an N–qubit system $|\Psi_j^{\pm}\rangle$ is defined as follows:

$$|\Psi_j^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|0j\rangle \pm |1\bar{j}\rangle\right) \quad (0 \le j \le 2^{N-1}-1), \quad (2)$$

where $j \equiv \sum_{i=2}^{N} j_i 2^{i-2}$ for the binary number j_i ($= 0, 1$), $|0j\rangle \equiv |0\rangle_1 \otimes \bigotimes_{i=2}^{N} |j_i\rangle_i$ and $|1\bar{j}\rangle \equiv |1\rangle_1 \otimes \bigotimes_{i=2}^{N} |1-j_i\rangle_i$. The symbol $\bar{j}$ means a bit–flip of j: $\bar{j} = 2^{N-1} - 1 - j$. We write the computational basis for the ith qubit as $|0\rangle_i$ and $|1\rangle_i$ (${}_i\langle 0|0\rangle_i = 1$, ${}_i\langle 1|1\rangle_i = 1$, and ${}_i\langle 0|1\rangle_i = 0$). The subscription $i(= 1, 2, \ldots N)$ is the label of the qubit. We can easily find the generalized GHZ states are the elements of an orthonormal basis of the Hilbert space corresponding to the N–qubit system. Note that the convention of generalized GHZ states (2) is slightly different from the corresponding one in Ref. [16, 17, 18, 19], but such a difference doesn't matter in our discussion.

We summarize the several useful properties of ρ_N. The compact consequences for partial transposition with respect to any bipartition are known [16, 17, 18, 19]. First, ρ_N has positive partial transposition (PPT) with respect to a bipartition k if and only if $\Delta \le 2\lambda_k$, where $\Delta = |\lambda_0^+ - \lambda_0^-|$. On the other hand, ρ_N has negative partial transposition (NPT) with respect to a bipartition k if and only if $\Delta > 2\lambda_k$. Furthermore, the authors in Ref. [17, 18] proved the theorems about multiparticle entanglement. Among them, we explain an important one [17]. We concentrate on two qubits, for example the ith and jth qubits, in an N–qubit system. Let us consider all possible bipartitions, $\mathcal{P}_{ij}$ under which the ith and jth qubits belong to different parties. The theorem is that ρ_N has NPT with respect to $^{\forall}k \in \mathcal{P}_{ij}$ if and only if the maximal entangled states between the ith and jth qubits can be distilled.

The most important result in Ref. [16, 17] is that an arbitrary density matrix, ρ of an N–qubit system, can be transformed into ρ_N by local operations, and local operations cannot increase entanglement. Accordingly, if ρ_N is an entangled state with respect to a bipartition, ρ is also such a state. Moreover, according to the theorem explained at the end of the above paragraph, if ρ_N has NPT with respect to $^{\forall}k \in \mathcal{P}_{ij}$, the maximal entangled state between the ith and jth qubits can be distilled from ρ_N. Then, one should be able to distill the maximal entangled state between such qubits from ρ. This result implies that one can know the sufficient condition for the distillability of ρ for an arbitrary N. Note that, through the PPT criterion, one can only obtain the necessary condition for the distillability in an N-qubit system when $N > 2$ [6].

Under the local operations, the coefficients $\lambda_0^{\pm}$ and λ_j of ρ_N are given by the following relations:

$$\lambda_0^{\pm} = \langle\Psi_0^{\pm}|\rho|\Psi_0^{\pm}\rangle, \quad 2\lambda_j = \langle\Psi_j^+|\rho|\Psi_j^+\rangle + \langle\Psi_j^-|\rho|\Psi_j^-\rangle. \quad (3)$$

Consequently, one can systematically treat the evaluation of multiparticle entanglement as a task for bipartite entanglement, because it is only necessary to calculate some specific matrix elements of ρ. In addition, this point will be useful for investigating entanglement in experiments [19].

III. LOCAL OPERATIONS IN THE DC METHOD

As shown in the previous section, one can readily evaluate the information of multiparticle entanglement by the DC method. However, the desired information about entanglement isn't always obtained. Let us illustrate such an impractical case by an example in a 2–qubit system. One can easily find that, by the PPT criterion, the following density matrix has NPT (i.e., entangled):

$$\rho_f = \frac{1}{2}|\Psi_0^+\rangle\langle\Psi_0^+| + \frac{1}{4}|\Psi_1^+\rangle\langle\Psi_1^+| + \frac{1}{4}|\Psi_1^-\rangle\langle\Psi_1^-| + \frac{1}{4}\big(|\Psi_1^+\rangle\langle\Psi_1^-| + |\Psi_1^-\rangle\langle\Psi_1^+|\big). \quad (4)$$

One needs only to calculate Δ and $2\lambda_1$ to apply the DC method to a 2–qubit system. According to Eq. (3), one can readily obtain the following results for ρ_f: $\Delta = \frac{1}{2}$ and $2\lambda_1 = \frac{1}{2}$. Then, it is not possible to determine whether ρ_f is entangled or not, because $\Delta = 2\lambda_1$.

We will show that the above problem should be attributed to the decrease of entanglement by local operations in the DC method. Let us explain Dür and Cirac's

explicit expressions to clarify this point. The local operations in the DC method are sequence of the following three steps. First, we perform the following probabilistic unitary operator on an arbitrary density operator of an N–qubit system:

$$\mathcal{L}_1\rho = \frac{1}{2}\rho + \frac{1}{2}W_1\rho W_1^\dagger, \tag{5}$$

where $W_1 = \bigotimes_{i=1}^N \sigma_x^{(i)}$ and $\sigma_x^{(i)} = |0\rangle_i\langle 1| + |1\rangle_i\langle 0|$. Note that

$$\rho = \sum_{j,\,j'=0}^{N} \Big(\mu_{jj'}^{++}|\Psi_j^+\rangle\langle\Psi_{j'}^+| + \mu_{jj'}^{+-}|\Psi_j^+\rangle\langle\Psi_{j'}^-| + \mu_{jj'}^{-+}|\Psi_j^-\rangle\langle\Psi_{j'}^+| + \mu_{jj'}^{--}|\Psi_j^-\rangle\langle\Psi_{j'}^-| \Big), \tag{6}$$

where $\mu_{jj'}^{\sigma\sigma'}$s are the matrix elements of ρ for the generalized GHZ states (σ, $\sigma' = \pm$). As a result of this operation, the terms corresponding to $|\Psi_j^+\rangle\langle\Psi_{j'}^-|$ and $|\Psi_j^-\rangle\langle\Psi_{j'}^+|$ are vanishing because $W_1|\Psi_j^\pm\rangle = \pm|\Psi_j^\pm\rangle$.

The following probabilistic unitary operators are necessary for the second step:

$$\mathcal{L}_l\rho = \frac{1}{2}\rho + \frac{1}{2}W_l\rho W_l^\dagger \qquad (l = 2,\, 3,\, \ldots,\, N), \tag{7}$$

where $W_l = \sigma_z^{(1)} \otimes \sigma_z^{(l)}$ and $\sigma_z^{(i)} = |0\rangle_i\langle 0| - |1\rangle_i\langle 1|$. Equation (7) is a local operation with respect to the first and lth qubit. Note that we abbreviate the identity operators for the other qubits in W_l. In the second step, we perform $\prod_{l=2}^N \mathcal{L}_l$ on the result of the first step. By this operation, the terms corresponding to $|\Psi_j^\pm\rangle\langle\Psi_{j'}^\pm|$ ($j \neq j'$) are vanishing because $W_l|\Psi_j^\pm\rangle = (-1)^{j_l}|\Psi_j^\pm\rangle$. In this stage, the resultant state is a diagonal form with respect to the generalized GHZ states.

Finally, we perform the local random phase–shift, $\mathcal{L}_r$ on the result of the second step:

$$\mathcal{L}_r\rho = \prod_{i=1}^N \left(\int_0^{2\pi} \frac{d\phi_i}{2\pi} \right) 2\pi\,\delta(\Phi - 2\pi)\, R_\phi\, \rho\, R_\phi^\dagger, \tag{8}$$

where $R_\phi = \bigotimes_{i=1}^N R^{(i)}(\phi_i)$, $R^{(i)}(\phi_i)|0\rangle_i = e^{i\phi_i}|0\rangle_i$, $R^{(i)}(\phi_i)|1\rangle_i = |1\rangle_i$, and $\Phi = \sum_{i=1}^N \phi_i$. Note that $\mathcal{L}_r\,|\Psi_0^\pm\rangle\langle\Psi_0^\pm| = |\Psi_0^\pm\rangle\langle\Psi_0^\pm|$ and $\mathcal{L}_r\,|\Psi_j^\pm\rangle\langle\Psi_j^\pm| = \frac{1}{2}(|0j\rangle\langle 0j| + |1\bar{j}\rangle\langle 1\bar{j}|)$ ($j \neq 0$). After the final step, we can find that the resultant state is equivalent to Eq. (1).

Now, let us go back to Eq. (4). We only need to perform $\mathcal{L}_1$ on ρ_f to transform it into the form of Eq. (1): $\mathcal{L}_1\rho_f = \frac{1}{2}|\Psi_0^+\rangle\langle\Psi_0^+| + \frac{1}{4}|\Psi_1^+\rangle\langle\Psi_1^+| + \frac{1}{4}|\Psi_1^-\rangle\langle\Psi_1^-|$. Obviously, the resultant state is separable. It implies that the entanglement decreases by the local operation $\mathcal{L}_1$. In the subsequent section, we will characterize the class of the quantum states in a 2–qubit system whose entangled property is not obtained by the DC method due to its decrease by the local operations.

IV. LIMITATION OF THE DC METHOD IN 2–QUBIT SYSTEMS

We attempt to reveal the class of the quantum states whose entangled property is not obtained by the DC method. In this section, we focus on the case $N = 2$ because its entanglement structure is well known.

Let us first introduce an important quantity for our consideration:

$$\mathcal{F}(\rho) = \max_{U,\,V} \langle\Psi_0^+|(U \otimes V)\,\rho\,(U \otimes V)^\dagger|\Psi_0^+\rangle, \tag{9}$$

where U and V are unitary operators on the Hilbert spaces for the first and second qubits, respectively. Equation (9) is called a fully entangled fraction [7, 10].

We show that the value of a fully entangled fraction plays an important role in determining whether the DC method works or not. According to the DC method, the sufficient condition for an entangled state in a 2–qubit system is $\Delta > 2\lambda_1$. Using $\mathrm{tr}\rho = 1$ and Eq. (3), we readily obtain the following relation:

$$\Delta > 2\lambda_1 \iff \langle\Psi_0^+|\rho|\Psi_0^+\rangle > \frac{1}{2} \quad \text{or} \quad \langle\Psi_0^-|\rho|\Psi_0^-\rangle > \frac{1}{2} \tag{10}$$

The right–hand side of Eq. (10) implies $\mathcal{F}(\rho) > \frac{1}{2}$. Note that $|\Psi_0^-\rangle = (I^{(1)} \otimes \sigma_z^{(2)})|\Psi_0^+\rangle$, where $I^{(1)} = |0\rangle_1\langle 0| + |1\rangle_1\langle 1|$. Summarizing the above argument, we obtain the following statements:

$$\Delta > 2\lambda_1 \implies \mathcal{F}(\rho) > \frac{1}{2}, \tag{11}$$

or

$$\mathcal{F}(\rho) \leq \frac{1}{2} \implies \Delta \leq 2\lambda_1. \tag{12}$$

Accordingly, we obtain the following conclusion. Let us consider the density matrix in a 2–qubit system which has NPT; it is an entangled state. However, if its fully entangled fraction is less than or equal to $\frac{1}{2}$, then it is not possible to determine whether such a state is entangled or not by the DC method. Actually, Eq. (4) is just such an example.

Next, we investigate the density matrix ρ whose fully entangled fraction is greater than $\frac{1}{2}$. In general, the condition $\mathcal{F}(\rho) > \frac{1}{2}$ does not imply $\langle\Psi_0^\pm|\rho|\Psi_0^\pm\rangle > \frac{1}{2}$. However, the following statement is always true:

$$^\exists\tilde{U} \otimes \tilde{V} \quad \text{s.t.} \quad |\Psi_0^+\rangle = \tilde{U} \otimes \tilde{V}|\tilde{\psi}\rangle, \tag{13}$$

where $\tilde{U}$ and $\tilde{V}$ are unitary operators on the Hilbert spaces for the first and second qubits, respectively, and $|\tilde{\psi}\rangle$ is the maximally entangled state that satisfies $\langle\tilde{\psi}|\rho|\tilde{\psi}\rangle = \mathcal{F}(\rho)$. Consequently, using the above local unitary operator, we obtain

$$\langle\Psi_0^+|\tilde{\rho}|\Psi_0^+\rangle > \frac{1}{2}, \tag{14}$$

$$\tilde{\rho} = (\tilde{U} \otimes \tilde{V})\,\rho\,(\tilde{U} \otimes \tilde{V})^\dagger. \tag{15}$$

According to Eqs. (11) and (14), we obtain the following statement:

$$\mathcal{F}(\rho) > \frac{1}{2} \implies {}^{\exists}\tilde{U}\otimes\tilde{V} \quad \text{s.t.} \quad \tilde{\Delta} \equiv |\tilde{\lambda}_0^+ - \tilde{\lambda}_0^-| > 2\tilde{\lambda}_1, \tag{16}$$

where $\tilde{\lambda}_0^\pm = \langle\Psi_0^\pm|\tilde{\rho}|\Psi_0^\pm\rangle$ and $2\tilde{\lambda}_1 = \langle\Psi_1^+|\tilde{\rho}|\Psi_1^+\rangle + \langle\Psi_1^-|\tilde{\rho}|\Psi_1^-\rangle$. The local unitary transformed state $\tilde{\rho}$ is entangled if $\tilde{\Delta} > 2\tilde{\lambda}_1$; one can obtain the entangled property of $\tilde{\rho}$ by the DC method. On the other hand, the original density matrix ρ is related to $\tilde{\rho}$ through the local unitary operator $\tilde{U}\otimes\tilde{V}$ from Eq. (15); $\tilde{\rho}$ is equivalent to ρ with respect to entanglement. Therefore, one can obtain the entangled property of a density matrix whose fully entangled fraction is greater than $\frac{1}{2}$ by the DC method with a suitable local unitary operator. Let us show an example for such a case. We consider a Bell–diagonal state. Such a state is defined by as follows:

$$\rho_{\rm BD} = \sum_{j=0}^{1} \left(\mu_j^+ |\Psi_j^+\rangle\langle\Psi_j^+| + \mu_j^- |\Psi_j^-\rangle\langle\Psi_j^-|\right), \tag{17}$$

where $\mu_j^\pm \geq 0$ and $\sum_{j=0}^{1}(\mu_j^+ + \mu_j^-) = 1$. Note that our example in Ref. [24] was a special case of Eq. (17). We can show that $\rho_{\rm BD}$ has NPT if and only if

$$|\mu_0^+ - \mu_0^-| > \mu_1^+ + \mu_1^- \quad \text{or} \quad |\mu_1^+ - \mu_1^-| > \mu_0^+ + \mu_0^-. \tag{18}$$

According to $\mathrm{tr}\,\rho_{\rm BD} = 1$ and Eq. (18), if one of μ_j^σs ($\sigma = \pm$) is at least greater than $\frac{1}{2}$, then $\rho_{\rm BD}$ has NPT, and vice versa. In addition, we easily obtain the following relation:

$$\mathcal{F}(\rho_{\rm BD}) = \max_{\sigma=\pm,\, j=0,\,1} \mu_j^\sigma. \tag{19}$$

Then, if $\rho_{\rm BD}$ is entangled, $\mathcal{F}(\rho_{\rm BD})$ is greater than $\frac{1}{2}$. In this case, we can obtain the information of the entanglement for the Bell–diagonal state in a 2–qubit system by the DC method with a suitable local operator. Note that one only needs to use $\mathcal{L}_r$ to transform $\rho_{\rm BD}$ into ρ_N.

Finally, we consider whether, through the DC method with appropriate local unitary operators, we can obtain the entangled property of the quantum state whose fully entangled fraction is less than or equal to $\frac{1}{2}$. It should be noted that the converse statement of Eq. (16) can be easily shown. Therefore, we conclude that one never obtains the entangled property for a density matrix whose fully entangled fraction is less than or equal to $\frac{1}{2}$ by the DC method, even if one uses local unitary operators.

In summary, we have completely classified the states in 2–qubit systems whose entangled property is not obtained by the DC method, or by the DC method with local unitary operators. The limitation of the method is determined by the value of the fully entangled fraction. If it is greater than $\frac{1}{2}$, we can always obtain the desired information on entanglement by the DC method with suitable local unitary operators. Otherwise, we never obtain it. The impracticality of the DC method is attributed to the decrease of entanglement by the local operations. Note that the Bell–diagonal state is entangled if $\mathcal{F}(\rho_{\rm BD}) > \frac{1}{2}$. Moreover, we can easily find $\mathcal{L}_1\,\rho_{\rm BD} = \mathcal{L}_2\,\rho_{\rm BD} = \rho_{\rm BD}$ and $\mathcal{L}_r\rho_{\rm BD} = \rho_N$. Accordingly, in 2–qubit systems, the crucial decrease of entanglement occurs in $\mathcal{L}_1$ and $\mathcal{L}_2$.

V. SUMMARY

We have shown that one cannot always obtain an entangled property by the DC method, even in 2–qubit systems. The most important quantity in our discussion is a fully entangled fraction. One can never determine whether a quantum state is entangled or not through the DC method, if the fully entangled fraction is less than or equal to $\frac{1}{2}$. On the other hand, one can make such a determination by the DC method with suitable local unitary operators, if the fully entangled fraction is greater than $\frac{1}{2}$.

The impracticality of the DC method is attributed to the decrease of entanglement by the local operations. Actually, from Eqs. (4) and (5), we have easily shown that $\mathcal{L}_1\rho_f$ is separable, even if ρ_f is entangled. The Bell–diagonal state (17) is invariant under $\mathcal{L}_1$ and $\mathcal{L}_2$; we only need to use $\mathcal{L}_r$ for transforming it into the form of Eq. (1). In addition, the Bell–diagonal state which is entangled has a fully entangled fraction greater than $\frac{1}{2}$. Therefore, the crucial decrease of entanglement for examining it by the DC method occurs in $\mathcal{L}_1$ and $\mathcal{L}_2$ in 2–qubit systems.

Finally, we would like to comment on the case of multiqubit systems. The DC method has been proposed as a systematic estimation of multiparticle entanglement. Therefore, it is necessary to study the limitation of the method in N–qubit systems when $N > 2$. However, the situation will be more complicated in this case. Nevertheless, the results in this letter can hint at a solution. Namely, we will consider the following question: (i) Is it possible to obtain the entangled property of Bell–diagonal states in N–qubit systems,

$$\rho_{\rm BD} = \sum_{j=0}^{2^{N-1}-1} \left(\mu_j^+ |\Psi_j^+\rangle\langle\Psi_j^+| + \mu_j^- |\Psi_j^-\rangle\langle\Psi_j^-|\right), \tag{20}$$

by the DC method with suitable local unitary operators? (ii) How are fragile quantum states with respect to entanglement, for example ρ_f, under the local operations characterized in N–qubit systems? We think the above questions are related to the decrease of quantum entanglement under local operations and decoherence. In addition, our examination of the above questions will lead to the understanding of the structure of quantum states in N–qubit systems.

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